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Representations of M_{12}*

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We introduce two families of lattices for the symmetric group and use them to build rank 10 and rank 45 lattices for the groups $2M_{12}2$ and M_{12}, respectively.

1. THE MESSIAH COMPLEX

In this section we describe the basic properties of the *Messiah Complex*, two related families of lattices which afford irreducible integral representations of the symmetric group.

The lattice of *thorns* consists of all zero-sum graphs on n vertices ($n \geq 4$), as follows. The underlying space has dimension $\binom{n}{2}$ and admits an orthonormal base $\{e_{12}, e_{13}, ..., e_{n-1,n}\}$ with $e_{ji} = -e_{ij}$. However, lattice vectors $\sum \alpha_{ij} e_{ij}$ satisfy the conditions

$$\begin{gathered} \alpha_{12} + \alpha_{13} + \cdots + \alpha_{1n} = 0, \\ \vdots \\ \alpha_{n1} + \alpha_{n2} + \cdots + \alpha_{n,n-1} = 0, \end{gathered}$$

where $\alpha_{ij} \in \mathbf{Z}$. Since only $n-1$ of these n conditions are independent, the lattice has rank $\binom{n-1}{2}$.

We depict a lattice vector as a zero-sum graph on n vertices by letting α_{ij} denote the multiplicity of a directed edge from vertex i to vertex j. The coordinate conditions ensure that the sum of the multiplicities of edges entering any vertex is equal to the sum of the multiplicities of edges leaving that vertex.

The $\pm\binom{n}{3}$ minimal vectors of the thorn lattice are of the form $e_{ij} + e_{jk} + e_{ki}$ as in Diagram 1. Henceforth this vector will be called the *thorn* T_{ijk}. A useful integral base of this lattice is $\{T_{1ij}\}_{2 \leq i < j \leq n}$; clearly $T_{ijk} = T_{1ij} + T_{1jk} + T_{1ki}$.

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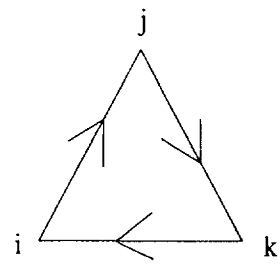


The thorn T_{ijk}.

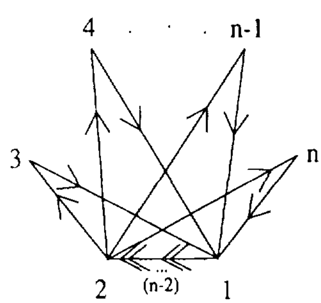
Closely related to the thorn lattice is the lattice of *crowns*. A crown C_{ij} is defined to be the sum of the $n-2$ thorns that contain the directed edge from i to j; for example, $C_{12} = T_{123} + T_{124} + \cdots + T_{12n}$ is pictured in Diagram 2. The lattice they generate is called the crown lattice.

If we choose as an integral base of this lattice the crowns C_{ij}, $2 \leqslant i < j \leqslant n$, then the crown and thorn lattices are scaled duals: $C_{ij} \cdot T_{1kl} = n\, \delta_{ik}\, \delta_{jl}$. Since all inner products in the crown lattice are divisible by n, we define the crown lattice inner product to be the natural inner product divided by n. Thus the $\pm\binom{n}{2}$ crowns have norm $n-2$ and two distinct crowns have inner product 0 or ± 1.

The automorphism group of the crown and thorn lattices on n vertices is $2 \times S_n$, generated by negation and vertex permutation. It is easily shown that these lattices afford absolutely irreducible representations.

2. A Lattice for $2M_{12}2$

The Messiah Complex simplifies the construction of lattices whose automorphism groups are known to contain S_n or large subgroups. The results are particularly appealing when the subgroup of S_n acts irreducibly.



The crown C_{12}.

This is the case for a 10-dimensional representation of $2M_{12}2$ over $\mathbf{Q}(\sqrt{-2})$. The restriction of this representation to a subgroup isomorphic to S_6 can be realized by a crown lattice on six vertices; since this S_6 subgroup is the stabilizer in M_{12} of a special hexad H_0, we label the vertices by the six points of the hexad. We call this lattice $\mathscr{C}_6^{H_0}$.

Furthermore, by adding six more vertices and labeling them by the points complementary to H_0, we obtain a natural action of M_{12} on 12 vertices. A typical element $g \in M_{12}$ then sends the vectors of $\mathscr{C}_6^{H_0}$ to vectors in $\mathscr{C}_6^{g(H_0)}$, the lattice generated by crown vectors on the six points of the special hexad $g(H_0)$. Naturally we seek a well-defined formula for converting crown vectors in an arbitrary hexad base into vectors of our original lattice $\mathscr{C}_6^{H_0}$—if this formula preserves inner products, we will have constructed a rank 10 lattice with a surjection

$$\Pi\colon \mathscr{A} \to M_{12}$$

for the subgroup $\mathscr{A}$ of lattice automorphisms generated by vertex permutation.

Remark. The character of the representation of $2M_{12}2$ remains an irreducible modular character for all primes; a well-known result of Brauer shows that the p-local Schur indices are all 1. Since the Frobenius–Schur indicator is 0, the representation can be realized over $\mathbf{Q}(\sqrt{-2})$. Furthermore, the irreducibility of the modular characters means that the lattice must be even and unimodular, but then the theta series of the lattice can be shown to be $1 + 3960q^4 + 168{,}960q^6 + \cdots$.

Notation. The zero-sum graphs on six vertices will be pictured in the following manner. A lattice vector $\sum \alpha_{ij} e_{ij}$ will have its coefficients α_{ij} denoted by the symbols $\rightarrowtail$ $(=1)$ and $\blacktriangleright$ $(=\sqrt{-2})$. For example, Diagram 3 depicts the vector $-e_{12} - e_{13} + (1+\sqrt{-2})\,e_{14} - e_{15} + (2-\sqrt{-2})\,e_{16} - e_{26} - (2+\sqrt{-2})\,e_{34} + (1+\sqrt{-2})\,e_{36} - e_{46} - e_{56}$. This vector has norm 4—recall that the inner product for a crown lattice on six vertices is the natural inner product divided by 6.

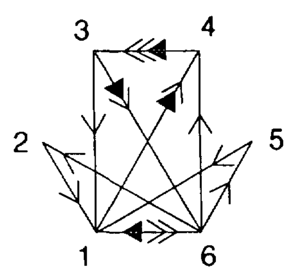


DIAGRAM 3

According to the remark above, our lattice contains 3960 minimal vectors: these are the 30 crown vectors in each of the 132 crown lattices $\mathscr{C}_6^H$, where H is any special hexad. The minimal vectors are therefore given "2 + 4" names $ab|cdef$ which identify the crown vector C_{ab} in the crown lattice on the six points of the special hexad $abcdef$. Of course, since the representation is of the group $2M_{12}$, we can describe the action of M_{12} only modulo ± 1; we therefore no longer distinguish between $ab|cdef$ and $ba|cdef$.

However, we still need to specify how the vector $ab|cdef$ can be converted into a vector in the $\mathscr{C}_6$ of any given hexad. Listed in Table I are 13 rules—one for each orbit of the original $2 \times S_6$ subgroup—which suffice to transform any minimal vector into a particular hexad base.

The first column displays the lattice vector when viewed in the crown lattice of the fixed hexad $abcdef$. The second column gives the $2+4$ name for the vector, where g, h, i, j, k, l represent the remaining six letters in an order which is restricted only by the condition that the permutations in column 3 lie in M_{12}. This order will vary from case to case. Permutations of shape $1^4 2^4$ are used whenever possible; these are the easiest M_{12} elements to generate since the four transpositions are the different pairs that complete the four fixed points into special hexads. The final column gives the number of lattice vectors in the orbit of the $2 \times S_6$ group of negation and vertex permutation.

The instructions are reversible—given any minimal vector and the hexad base, we can recover the $2+4$ name for the vector and its negative.

With all the vectors written in the form $ab|cdef$, the action of an element of $2M_{12}$ on any vector is easily computed to within sign; and once the particular image of a single vector is chosen, the images of other vectors are determined by their inner products with previously determined vectors. The lattice affords a faithful representation of the double cover $2M_{12}$, since elements of shape 2^6 lift to automorphisms of order 4.

The lattice vectors in a fixed hexad base are characterized as zero-sum graphs on six vertices with edge multiplicities $\in \mathbf{Z}[\sqrt{-2}]$ subject to three rules:

1. The sum over any crown is divisible by 6.
2. The sum over any directed square is divisible by $2+\sqrt{-2}$.
3. Any *2-thorn sum*—obtained by taking the sum over a thorn of the 6 vertices, multiplying it by $1-\sqrt{-2}$, and adding to it the sum over the thorn (in either direction) of the complementary three vertices—is divisible by $2-2\sqrt{-2}$.

Finally, we note that Orbits 1 and 2 of Table I both contain 15 pairs of vectors that generate a crown lattice, and this set of 30 pairs forms a

TABLE I

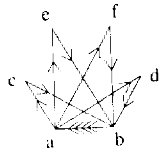
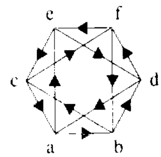
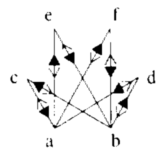
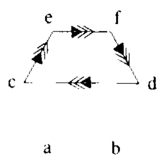
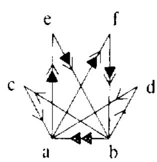
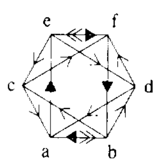
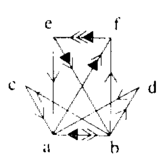
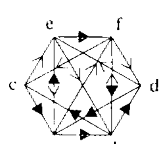
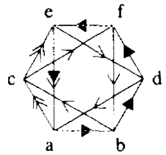
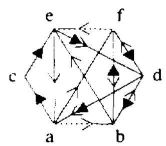
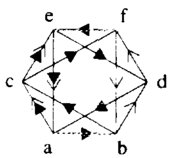
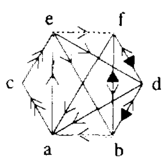
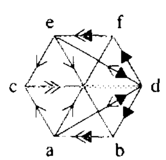
	$ab \mid cdef$		(30)
	$gh \mid ijkl$	$(af)(be)(cd)(gh)$	(30)
	$gh \mid cdef$	$(ab)(cd)(ef)(gh)$	(90)
	$ab \mid ghij$	$(ab)(cf)(de)(kl)$	(90)
	$ef \mid abgh$	$(ab)(cd)(ef)(gh)$	(180)
	$gh \mid cdij$	$(ab)(cd)(ef)(gh)$ $(aefb)(gkhl)$	(180)
	$gh \mid abij$	$(ab)(cd)(ef)(kl)$ $(afeb)(gkhl)$	(360)
	$ab \mid efgh$	$(ae)(bf)(cd)(gh)$	(360)

TABLE I—*Continued*

	$dg \mid bcfh$	$(ae)(bf)(cd)(gh)$ $(abdfe)(gijkl)$	(720)
	$dg \mid chij$	$(abdfec)(hij)(kl)$	(720)
	$gh \mid acei$	$(abcfed)(jkl)(gh)$	(240)
	$ae \mid cghi$	$(ace)(bfd)(ghi)$	(240)
	$cg \mid aehi$	$(ae)(bf)(hi)(jk)$ $(ace)(bfd)(ghi)$	(720)

bipartite graph if we connect two pairs when their representatives have inner product $\pm\sqrt{-2}$. Clearly these 60 vectors generate a lattice whose automorphism group is a maximal subgroup $2\,\mathrm{Aut}(A_6)$ of $2M_{12}$.

But these facts are equally true for the 30 pairs of the form $ab \mid ****$. The $2+12+16$ such vectors in Orbits 1, 5 and 12 generate one lattice isometric to a crown lattice, and the $6+24$ vectors in Orbits 4 and 8 generate another. The resulting isometry between these sublattices of 60 norm 4 vectors extends to a lattice automorphism that normalizes the subgroup $2M_{12}$ of vertex permutations. This automorphism projects to an outer automorphism of M_{12} which interchanges the set of subgroups that stabilize a pair of complementary hexads and the set of subgroups that stabilize a duad.

3. A. Lattice for M_{12}

The only representations of M_{24} which remain irreducible for all five Mathieu groups are two algebraically conjugate representations of degree 45. The character of either of these representations, when restricted to the smallest Mathieu group M_{11}, is the character of the crown and thorn lattices on eleven points. In this section, we show how the Messiah Complex leads directly to a lattice affording the irreducible 45-dimensional representation of M_{12}. Naturally, we begin with the action of M_{11} on a crown lattice with eleven vertices and adjoin a new vertex to obtain a more symmetric lattice.

As in the previous section, we suppose that M_{12} is the automorphism group of a system of 132 hexads on $(\infty, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X)$; the M_{11} we use is the stabilizer of the point ∞. Thus the crown lattice we start with has 11 vertices labeled $(0, 1, ..., X)$ and is generated by ± 55 crown vectors of norm 9. This lattice is denoted $\mathscr{C}_{11}^{\infty}$ to emphasize that it is the crown lattice on the eleven points excluding ∞. The stabilizer of a crown C_{ab} is the subgroup $M_9 \simeq 3^2 : Q_8$ of M_{11} that fixes the pair (a, b) pointwise, while the subgroup either fixing or negating the crown is $M_9 : 2$, the stabilizer of the duad (ab).

When a twelfth point labeled ∞ is added, an element $g \in M_{12}$ acting by vertex permutation sends the crown lattice $\mathscr{C}_{11}^{\infty}$ to the crown lattice $\mathscr{C}_{11}^{g(\infty)}$. We must therefore find a rule to display the vectors of any one of the twelve crown lattices as vectors of another.

The character table of M_{12} reveals that the one-dimensional space fixed by M_9 is also fixed by the group $M_9 : 3$ and negated by the rest of the maximal subgroup $M_9 : S_3$. But this means that the vector C_{ab}^{c} (the crown vector C_{ab} in the crown lattice $\mathscr{C}_{11}^{c}$ of the eleven points excluding c) can be renamed C_{bc}^{a} or C_{ca}^{b}. And the rule for displaying this vector in the space of the crown lattice $\mathscr{C}_{11}^{d}$ is reduced to determining two constants, since we now know precisely how the $Q_8 : S_3 \simeq 2S_4$ group that fixes the point d and the triad (abc) must act on the vector C_{ab}^{c}.

The renaming rule works as follows: the $2S_4$ group of the previous paragraph has four subgroups of order 3 and so four elements of type $(abc)(efg)(hij)$. In particular, exactly eight other 3-cycles (efg), (hij), ... appear in permutations containing the 3-cycle (abc). If T^d denotes the sum of the eight thorns

$$T_{efg}^{d} + T_{hij}^{d} + \cdots$$

(a sum of thorns in the 10-space underlying $\mathscr{C}_{11}^{d}$), then the crown vector C_{ab}^{c} in the d-base is $-T_{abc}^{d} - 2T^{d}$.

We recall that the lattice inner product is the natural inner product divided by 11. And if the vector C^c_{ab} is now labeled (abc), the description of the lattice can be neatly summarized:

The ± 220 generating vectors are indexed by triads (abc) of twelve letters, with $(abc) = (bca) = -(bac)$.

The inner product rule for these vectors is

$$\begin{array}{llll} (abc)\cdot(def) = & \pm 9 \text{ if } |(abc)\cap(def)| = 3 & & \\ & \pm 1 & \cdots & 2 \\ & 0 & \cdots & 1 \\ & 0 & \cdots & 0,\ (abcdef) \text{ is a special hexad} \\ & \pm 2 & \cdots & 0,\ (abcdef) \text{ is a non-special hexad.} \end{array}$$

The rule for signs is obvious when $|(abc)\cap(def)| = 3$; for triads with intersection of cardinality 2, the inner product is negative precisely when the common pair appears in the same order (i.e., (abc) and (abd)); and when there are no common elements, the inner product is 2 if $(abc)(de*)(f**)$ is an element of M_{12} and -2 if $(abc)(d*e)(f**)$ is an element, where the triad def may need to be cyclically permuted to obtain an element of the required type.

The automorphism group of the lattice is $2 \times M_{12}$, consisting of negation and permutation of the 12 letters.